Optimum Tolerance Assignment to Yield Minimum Manufacturing Cost

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The problem considered is that of choosing approximate component tolerances in order to minimize mass production costs. The basic item considered is a unit with a single nominal design response. This unit has several components with given nominal design values such that the unit nominal response is as required. We assume that the components are in statistical control and that we can compute the statistical behavior of the response as a function of the assignment of component tolerances. Further, we assume that the cost and salvage value of a unit are known as a function of the assignment of component tolerances. We impose the restriction that the sum of the responses of n identical units in combination must be within a prescribed tolerance with probability $1 - \epsilon$. *We can then find a relation involving the tolerance limits on the sum of the responses, the rejection limits on the response of a single unit, the variance of the response of a single unit, and the probability* ϵ. *Using this relation, which effectively introduces the rejection rate as an additional variable, we then show how to assign component tolerances to minimize production costs. As an illustrative example we consider the design for production of an idealized lumped-constant delay line.*

I. INTRODUCTION

A valid area of investigation for the cutting of manufacturing costs in the mass production process lies in the assignment of tolerances. In this paper we examine a problem in that area. Consider the following fairly typical sequence of events: A piece of equipment is to be designed with a specified nominal response, for example, an amplifier with a specified nominal gain, or a logic gate with a specified nominal time delay. The circuit is designed and nominal values are assigned to the components of this piece of equipment so that it has the required nominal response. Next, this piece of equipment is to be mass produced, and

mass produced economically. One of the manifold problems which arises at this point is the assignment of tolerances to the various components of the piece of equipment. It is at this point in the design for production that the considerations in this paper enter.

The effect of component tolerances is to cause the response to deviate from the nominal in a statistical manner. A common approach to component tolerance assignment ignores the statistical behavior of the response deviation and bases the tolerance assignment on the "worst case" approach, i.e., the deviations from nominal for all components are assumed to act in concert to maximize the deviation from nominal of the response. This criterion corresponds to a very pessimistic viewpoint because, usually, the probability of such a simultaneous occurrence of worst values is extremely small. In fact, it is often so small that in a very practical sense it is zero. Within the past several years the statistical approach to assigning component tolerances, which makes use of the statistical nature of the response deviation, has been gaining in popularity. J. M. Juran[1] gives examples and several references to uses of statistical tolerancing. A fine case history of a statistical tolerancing approach is that of the design for production of the repeaters used in the Bell Systems L3 coaxial system.[2] In References 3 and 4 the particular problems of statistical distribution requirements and quality control requirements for the components of the L3 system are considered. In statistical tolerancing, in order that the deviation of the response be in control, it is necessary that the component manufacturing processes either be in control or sufficiently compensated so that they are effectively in control at all times. We will assume statistical tolerancing in this paper; thus, we are also forced to assume the restrictive implication that the component manufacturing processes are in control.

In any kind of tolerancing there are many possible component tolerance assignments for which the response tolerances are identical or reasonably so. The costs associated with the different component tolerance assignments, however, will not in general be the same.

For example, consider an R-C circuit. Assuming for the sake of this example that both resistors and capacitors come in truly uniform distributions it is obvious that the statistical behavior of the time constant ($\tau = RC$) will be the same if the resistor is from a 5 per cent distribution and the capacitor is from a 10 per cent distribution or if the resistor is from a 10 per cent distribution and the capacitor is from a 5 per cent distribution. However, the costs will generally be different.

The desired tolerance assignment is the least expensive tolerance assignment (of those tolerance assignments which engender identical

response tolerances). Pike and Silverberg[5] have considered this problem for linear, or approximately linear, (mechanical) systems using statistical tolerancing. They show how to adjust the component tolerances (actually the variances) to get maximum value for minimum cost.

Next let us discuss some characteristics of the particular type problem we wish to consider:

1.1 *Response Tolerances*

It very often happens that the deviation from nominal of the response of an individual piece of equipment — or unit as we shall call it henceforth for brevity — is relatively unimportant; the quantity which is important is the algebraic sum of the deviations of the responses from nominal of a combination of several units.* For example, the repeaters in the L3 system are in series and the primary requirement is that the sum of the gains compensate for the line loss plus or minus a small tolerance. Another example of this type is a string of several logic gates for which the total time delay must be less than some prescribed value. The sum requirement gives us considerably more latitude in the assignment of the response tolerances for the individual units because of the nature of a sum of random variables — for indeed, the deviation from nominal of the response is a random variable under statistical tolerancing.

1.2 *Rejection Rate*

A criterion which is often used to measure the efficiency and economy of a production process is the rejection rate. Completed or uncompleted units may be rejected for any number of reasons, but here we confine our attention to those units which are rejected solely because the deviation of their components from nominal is such that their responses are out of tolerance. That is, we ignore those units which must be rejected because of cold solder joints, flaws, broken leads, and a multitude of similar causes. Hereinafter, we shall use the term rejection rate to mean the fraction of completed units which have a response which is outside of tolerance but which are otherwise acceptable. The usual assumption is that the rejection rate must be small for an economical production process, however, we take the viewpoint that the rejection rate is an-

* We have chosen to use the hierarchy: components, units, combinations of units. This triad may be thought of as corresponding to any similarly ordered threesome in any hierarchy which may be more familiar to the engineer, e.g., raw materials, piece parts, subassemblies, assemblies, units of product, subsystems, and systems.

other variable which may be introduced in order to minimize production costs. Note that this implies 100 per cent testing on finished units and the consequent added cost thereof.

But then, what of the rejected units? The rejected units will have some salvage value. The salvage value for a rejected unit may range from a positive value which is a fairly large percentage of the cost of manufacturing a unit (such as would be the case if only a small additional charge were necessary to bring the unit into tolerance or if out of tolerance units could be selectively assembled), to a negative value (such as would be the case if the unit were a total loss and there was an additional charge to dispose of it). In the most general case the salvage value is a statistical quantity since its value might depend, for example, on how far out of tolerance the response is or what component or combination of components is the essential cause of the response being out of tolerance.

1.3 *Aim*

Before going to the analysis, I would like to indicate the tenor of this work. Certainly we are trying to decrease production cost by an intelligent assignment of tolerances. However, it is important to note that this assignment is made at a point in the production process immediately after the final circuit design and specification of nominal component values have been completed. At this stage only the rudiments of the projected manufacturing process are known since many final answers must await the assignment of component tolerances. Hence, the figures for the production costs and the salvage values are not known precisely and may be in fact only educated guesses; in addition, the distributions for some of the components may not be known precisely. And further, it would be uneconomical to get precise estimates of the figures for each and every possible combination of component tolerances which could reasonably be used in the production model since the number of such combinations can easily be enormous. Thus, since the cost figures are not known precisely, it would be so much wasted effort to make the rest of the analysis exact. The principal advantage to be gained from the following analysis is to eliminate all except, say, two or three possible combinations of component tolerances for ultimate consideration for the production model.

II. GENERAL STATEMENT OF THE PROBLEM

Let us denote the deviation from nominal of the response of a unit by x; x is a random variable. Let there be n units in combination;*

* Combination, as we use it here, implies only that (1) holds.

n is a fixed but arbitrary number. Let x_i be the deviation from nominal of the response of the ith unit, $i = 1, 2, \cdots, n$, and let the x_i's be independent. Let it be required that, for proper over-all operation, the algebraic sum of the random deviations of the n units in combination be constrained to lie between $\pm B$; i.e.,

$$| x_1 + x_2 + \cdots + x_n | \leqq B. \tag{1}$$

That is, $\pm B$ are the tolerance limits on the deviation from nominal of the response for the combination of n units.

As is usual in statistical tolerancing let us be willing to assume a small risk ϵ that the combination will not operate properly, i.e., that the sum (1) will exceed B. Thus,

$$\Pr(| x_1 + x_2 \cdots + x_n | > B) = \epsilon. \tag{2}$$

Next let us look at the assignment of component tolerances. The assignment variable is really the independent variable in a tolerancing problem. That is, let the unit which is to be manufactured have k components; number these components arbitrarily, $1, 2, \cdots, k$. Let component #j be available in r_j different tolerance distributions which are to be considered as candidates for possible use in the production model of the unit; number these tolerance distributions arbitrarily, $1, 2, \cdots, r_j$. Do the same for all components, $j = 1, 2, \cdots, k$.

For example, suppose that component #j is a resistor. Let the available tolerance distributions considered be 5 per cent resistors and 10 per cent resistors. Thus $r_j = 2$ and we can arbitrarily number the 5 per cent resistors as distribution #1 and the 10 per cent resistors as distribution #2.

A particular assignment of component tolerances can thus be characterized by the ordered set of numbers

$$(i_1, i_2, \cdots i_k) \tag{3}$$

which is to be understood to mean that component #1 comes from the i_1th distribution from the set of available distributions for component #1, component #2 comes from the i_2th distribution from the set of available distributions for component #2, and so forth. The number of different tolerance assignments can be very large since the assignments range over all possible combinations of available distributions.* Although in the above we have only considered a finite number of tolerance distributions for each component there is no reason, in principle, why one

* The number is $\prod_{j=1}^{k} r_j$.

or more of the components cannot come from a continuum of possible tolerance distributions.

The component tolerance assignment variable, (3), is unwieldy; let us replace it by a more manageable independent variable. To do so we argue as follows: Since the deviation from nominal of the response of a unit, x, is a random variable, as such, it is characterized by a probability distribution, say $D(x)$. But $D(x)$ also depends on which set of component tolerances is used since different assignments of component tolerances will, in general, manifest themselves in different statistical behaviors for the response. A measure of the distribution $D(x)$ is the variance of x — $\text{var } x = \sigma^2$. The quantity σ^2, or σ, is an excellent measure if all distributions $D(x)$ are normally distributed with mean zero, as we shall shortly assume is the case in our problem; otherwise, the aptness of σ diminishes as $D(x)$ departs from normal with mean zero. Thus we can make the new independent variable σ (or σ^2, whichever is more convenient) instead of (3). The range of σ is determined by considering all possible combinations of component tolerances and σ can only take on the discrete values determined by the possible combinations of component tolerances (if only a finite number of distributions is considered for each component). Note that, at this point, the correspondence between σ and the particular assignment of component tolerances is not necessarily one to one, (see example in Section I). A unique (or effectively unique) correspondence will come about naturally when we consider the costs, below.

Before considering the costs we must consider the rejection rate. In the introduction we defined the rejection rate as the fraction of the completed units which have responses outside of tolerance but which are otherwise acceptable. That is, if the tolerance limits on the unit response are $\pm b$,* then every unit which has a response deviating from the nominal response by more than $\pm b$ is to be rejected, i.e., reject all units such that $|x| > b$. Since the rejection rate is a variable, b is a variable which must be determined. The tolerance limit b is a function of three variables σ, B, and ϵ, and must be chosen to satisfy (2). Qualitatively, for fixed B and ϵ it is obvious that, in order to satisfy (2), as σ increases b must decrease and vice versa. We will obtain a quantitative relation later.

Finally we consider the manufacturing costs per unit. We distinguish two types, the *raw cost* and the *real cost*. The raw cost per unit is the amount of money which must be spent to manufacture one unit regardless of whether it has a response which is or is not within the tolerance

* We are only going to consider symmetrical distributions about the nominal, hence b is sufficient.

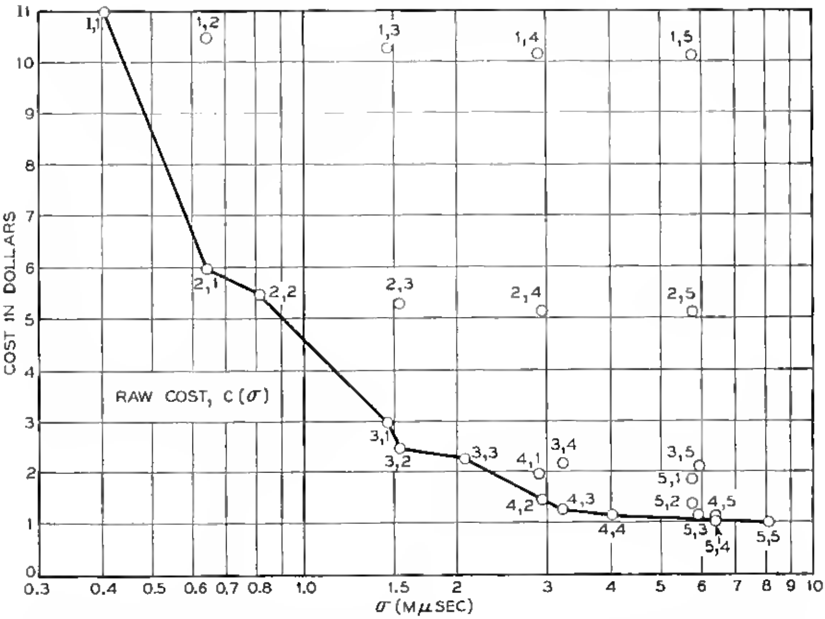

Fig. 1 — Raw cost for a single L-C section of a lumped constant delay line as a function of the standard deviation of the delay per section. (See Table II-8 for explanation of the code (i,j)).

limits determined by $\pm b$ and independent of any salvage value a unit may have. The real cost per unit is the raw cost plus the unsalvageable raw cost per rejected unit prorated among the units within tolerance. It is obvious that for a well behaved manufacturing process the raw cost should be a monotone decreasing function of σ. Furthermore, if two or more different component tolerance assignments give the same — or approximately the same — variance, σ^2, the assignment which should be chosen to correspond to that σ is the one which minimizes the raw cost. A better statement of the criterion for choosing the component tolerance assignments which make up the raw cost curve as a function of σ is that an assignment lies on the raw cost curve if there is no other assignment which has *both* a smaller (or as small) σ *and* a lower (or as low) cost.* A raw cost curve, $C(\sigma)$, is illustrated in Fig. 1, i.e., if

* Stated precisely, the points of the raw cost function as a function of σ, $C(\sigma)$, are determined as follows: Let us denote the component tolerance assignment variable, (3), by β; let the raw cost for each β be $C(\beta)$; let the variance of the response for each β be $[\sigma(\beta)]^2$; then the points of the raw cost curve are given by

$$C(\sigma) = \min C(\beta) \tag{4}$$

where (a) the minimum is taken over all β such that $\sigma(\beta) \leqq \sigma$, and (b) the only allowed values of σ are those such that there exists a corresponding β *and* $C(\beta) = C(\sigma)$, i.e., the "corners" of (4).

one plots the point (σ, C) — or equivalently (σ^2, C) — for each assignment of component tolerances then $C(\sigma)$ is the set of points which determine the polygonal curve which is the lower envelope of the set of points for all possible combinations of component tolerances. The set of points $C(\sigma)$ are connected for illustrative purposes only; $C(\sigma)$ exists only as a pointwise function (for component tolerances which do not come from a continuum of allowed tolerances).

Finally there is the salvage value for the rejected units. We denote the salvage value by $\alpha(\sigma)C(\sigma)$, i.e., α is the ratio of the salvage value to the raw cost. In the general case the salvage will be a function of σ, i.e., of the particular set of component tolerances, and it will also be a random variable which depends on x, the deviation of the response from nominal. We will retain the dependence of α on σ; but we will ignore the fact that it may be a random variable and take α as a constant for each σ. This constant value may be an expected value. The assumptions set forth on α are in accord with the aim set forth in the introduction, for, if the cost figures are not precise estimates, then certainly the salvage value as a distribution function cannot be known precisely. If we took α in all its generality, we would only succeed in cluttering up the analysis with functions and figures for which we could not possibly get realistic estimates. We can, however, reasonably expect to get a realistic estimate for the expected value of the salvage as a function of the component tolerance assignment, or equivalently σ. Along this same line of reasoning, in connection with the salvage value, we note that we assumed that if two different assignments of component tolerances give the same σ then the possible difference in their salvage values was to be ignored in choosing the assignment which determines the raw cost curve $C(\sigma)$. This assumption could possibly lead to a real cost which is higher than necessary since the salvage value is inherent in the real cost. However, the possibility of such an occurrence is doubtful, and if such an occurrence were suspected it could always be calculated as a special case.

The problem we want to solve is:

(a) given n, B, ϵ, as defined in the first two paragraphs of this section,

(b) given the raw cost as a function of σ, $C(\sigma)$,

(c) given the salvage as a function of σ, $\alpha(\sigma)$,

find the value of σ such that the real cost, $C^*(\sigma)$, is a minimum and find the tolerance limits on the unit response, $\pm b$.

We are able to solve this problem under restrictive but widely applicable conditions.

III. RELATIONSHIP BETWEEN b AND σ

We assume that the deviation of the response from nominal of a unit, x, is normally distributed with standard deviation σ. This is a realistic assumption if the components used in manufacturing the unit have independent random variations (not necessarily normally distributed) which in turn influence the response additively. We assume, further, that the mean of x is always zero which in turn implies that the mean does not shift significantly with change in σ, i.e., with change in the assignment of component tolerances, and further that the component manufacturing processes are in control.

Let $F(y)$ be the cumulative normal distribution function and $\varphi(y)$ the normal probability density function:

$$\begin{aligned} F(y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\, dt, \\ \varphi(y) &= \frac{1}{\sqrt{2\pi}} e^{-y^2/2}. \end{aligned} \tag{5}$$

If the tolerance limits on x are $\pm b$ then the probability that an individual unit will be rejected, i.e., the rejection rate, is

$$\text{Probability of rejection} = \text{rejection rate} = 2[1 - F(b/\sigma)]. \tag{6}$$

Since only the units which fall within the rejection limits $\pm b$ are to be used in the combination of n units, the probability density function for the acceptable units is

$$\psi(x) = \begin{cases} \dfrac{\varphi(x/\sigma)}{\sigma[2F(b/\sigma) - 1]}, & |x| < b \\ 0, & |x| > b \end{cases}. \tag{7}$$

We want next to find the distribution function for the random variable

$$\xi = x_1 + x_2 + \cdots + x_n, \tag{8}$$

where the x_i are independent and distributed according to (7). We assume that n is sufficiently large to apply the central limit theorem. Performing the necessary integration to find the variance of x distributed according to (7), we have that ξ is normally distributed with mean zero and variance

$$\sigma_\xi^2 = n\sigma^2 u(b/\sigma), \tag{9}$$

where

$$u(t) = \left[1 - \frac{2t\varphi(t)}{2F(t) - 1}\right], \qquad t > 0. \tag{10}$$

Note that $u(t) \leqq 1$.

Because of the above assumptions we can rewrite (2) as

$$F(B/\sigma_\xi) = 1 - \epsilon/2. \tag{11}$$

From tables for $F(y)$ we can find the standard normal deviate $r = B/\sigma_\xi$ for a given risk ϵ. Introducing r in (9) to eliminate σ_ξ we find

$$(B/\sigma)^2 = r^2 nu(b/\sigma). \tag{12}$$

Equation (12) gives the desired relationship among B, r (or ϵ), σ and b in order to satisfy (2) for an arbitrary value of σ. Note that if, in trying to satisfy (12), u turns out to be greater than one this simply means that although all units are accepted the probability that $|\sum x_i|$ exceeds B is still less than ϵ.

IV. REAL COST PER UNIT

The raw cost per unit, as we have defined it, does not include the penalty that must be paid for producing units which are outside of tolerance and therefore must be sent to salvage, nor does it include any salvage value the rejected units may have. Call the raw cost per unit

$$C = C(\sigma). \tag{13}$$

In addition to knowing the raw cost we must also know the salvage value of a rejected unit, i.e., a unit such that $|x| > b$. As before, we define the salvage value per unit to be

$$S = \alpha(\sigma)C(\sigma). \tag{14}$$

Here, α is a proportionality factor which will, in general, depend on σ. Obviously α is less than 1; on the other hand it may range downward thru negative values, e.g., if it costs additional money to dispose of a rejected unit.

Define the real cost per unit to be, as before,

$$C^* = C^*(\sigma). \tag{15}$$

The real cost is related to the raw cost and the salvage value as follows: If M units are produced in all and m of these M units must be rejected and sent to salvage because their responses are out of tolerance, then

$$C^*(\sigma) = \frac{1}{M - m}\left[MC(\sigma) - m\alpha(\sigma)C(\sigma)\right] \tag{16}$$

is the real cost per unit, i.e., C^* is the total raw cost for all units produced minus the salvage value of the unacceptable units all prorated among the acceptable units. For large M, m/M is the probability that a unit will fall outside of tolerance, i.e., it is the rejection rate, (6); hence

$$C^*(\sigma) = \left[\frac{1-\alpha}{2F(b/\sigma) - 1} + \alpha\right] C(\sigma). \tag{17}$$

By proper choice of σ, we want to minimize the function C^*.

V. MINIMIZATION OF THE REAL COST

In principle we could give functional forms for $C(\sigma)$ and $\alpha(\sigma)$ and then minimize $C^*(\sigma)$ by the usual analytical methods. However, one would rarely, if ever, know the functional form for either. Hence we go to a graphical method.

So that the necessary calculations may be carried out expeditiously we redefine some of the previously formulated functions. First, however, let us see exactly what is desired. We are given B, ϵ(or r) and n. We want to calculate $C^*(\sigma)$ throughout the range of interest of σ (or specifically, for a set of values of σ in the range of interest). After plotting $C^*(\sigma)$ we can pick off the minimum, or minimums, of C^*; we then need to calculate b for the minimum, or minimums. The calculation of $C^*(\sigma)$ and b for given B, r, n, σ, can be done stepwise:

1. From (12) calculate u.
2. From (10) calculate the implicitly defined variable $t = b/\sigma$ for u from step 1. (The correspondence between t and u is one-to-one since u is a strictly monotone* function of t). This step essentially gives us b.
3. From (17), using $t = b/\sigma$ from step 2, calculate $C^*(\sigma)$.

Now that we know exactly what is desired we can expedite the calculations. A convenient combination of the variables is

$$q = \frac{r\sqrt{n}}{B}\,\sigma = u^{-1/2}. \tag{18}$$

Since $u(t)$, defined by (10), is strictly monotone it can be inverted (numerically) to get

$$t = w(q). \tag{19}$$

Also, define the function

$$H(q) = \frac{1}{2F(w(q)) - 1} = \frac{1}{2F(t) - 1} \tag{20}$$

* The fact that u is strictly monotone will become obvious from the graph of the related function (19), Fig. 2.

TABLE I

$H(q)$ and $w(q)$ as functions of the argument q

q	H	w
1.01	1.002	3.11
1.02	1.004	2.86
1.03	1.007	2.69
1.04	1.010	2.57
1.05	1.013	2.48
1.06	1.017	2.40
1.07	1.020	2.33
1.08	1.024	2.26
1.09	1.028	2.21
*1.10	1.032	2.16
1.12	1.041	2.05
1.14	1.049	1.99
1.16	1.058	1.92
1.18	1.069	1.85
*1.20	1.077	1.80
1.25	1.10	1.68
1.30	1.13	1.58
1.35	1.16	1.49
1.40	1.19	1.42
1.45	1.22	1.35
*1.50	1.25	1.29
1.60	1.31	1.19
1.70	1.37	1.11
1.80	1.43	1.03
1.90	1.49	0.98
*2.00	1.56	0.92
2.50	1.90	0.72
3.00	2.25	0.59
3.50	2.60	0.50
4.00	2.95	0.44
4.50	3.30	0.39
*5.00	3.65	0.35
6.00	4.37	0.29
7.00	5.09	0.25
8.00	5.82	0.22
9.00	6.54	0.19
10.00	7.27	0.17

* Indicates a change in increment of the argument q

using t and w as defined by (19). Both of the functions H and w have the same independent variable, q. In terms of the functions w and H we have

$$C^*(\sigma) = \left[(1 - \alpha)H\left(\frac{r\sqrt{n}}{B}\,\sigma\right) + \alpha\right]C(\sigma), \tag{21}$$

and

$$b = \sigma \cdot w\left(\frac{r\sqrt{n}}{B}\,\sigma\right) \tag{22}$$

as the desired formulas. The functions $w(q)$ and $H(q)$ are tabulated in Table 1 and plotted in Fig. 2 for convenient use. We note for $q < 1$ that $t = w(q)$ is infinite; this simply expresses the fact that for σ sufficiently small the probability that $|\sum x_i|$ will exceed B is less than ϵ, and, hence, that the rejection limits are $\pm b = \pm\infty$, cf., remark about u, following (12).

In terms of H, the rejection rate (6) is

$$\text{Rejection rate} = 1 - 1/H. \tag{23}$$

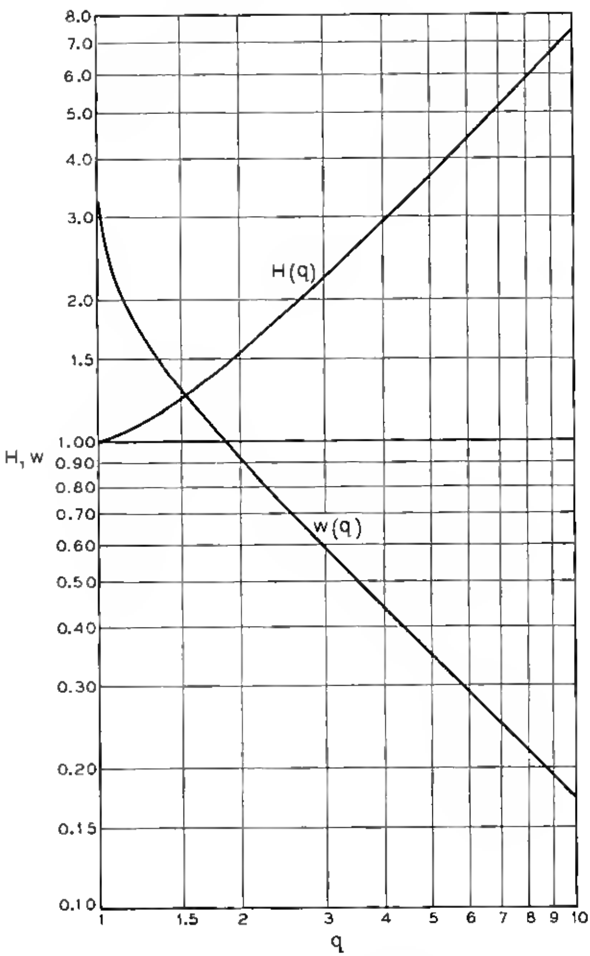

Fig. 2 — $H(q)$ and $w(q)$ as functions of the argument q.

VI. COMPARISON WITH OTHER CRITERIA

In the above we have tacitly assumed that each unit would be tested. Let us now consider the case in which this test is omitted, that is, at least as a test on 100 per cent of the units. One can still satisfy (2) by choosing the proper value for σ. Since no units are rejected the distribution for $\xi = \sum x_i$ is normal with mean zero and variance $\sigma_\xi^2 = n\sigma^2$. Hence (2) becomes

$$F(B/\sigma\sqrt{n}) = 1 - \epsilon/2. \tag{24}$$

Letting r be the standard normal deviate which satisfies (24) one has

$$\sigma = B/r\sqrt{n}, \qquad (b = \infty). \tag{25}$$

This is, of course, well known and is in use. In comparing the cost by this last method with the cost by the previous method, one must remember to take into account the cost of 100 per cent testing of units. The testing cost could easily swing the balance in favor of the no-test method.

Another criterion to consider is the zero risk case. Here, ϵ equals zero and the rejection limits are then given by $b = B/n$. It still remains to choose the optimum σ for the manufacturing process. Proceeding in the same manner as previously, one finds that the real cost is given by

$$C^*(\sigma) = \left[\frac{1 - \alpha}{2F(B/n\sigma) - 1} + \alpha\right] C(\sigma), \tag{26}$$

This can obviously be plotted as a function of σ and the minimum for C^* obtained graphically. Note that in this case the component distributions do not have to be in control to satisfy the tolerance limits $\pm B$; however, they must be reasonably in control to make (26) true.

VII. EXAMPLE

As an idealized example of the method described in the foregoing we consider the design for production of a lumped constant delay line. This example is meant to be strictly illustrative since we wish to concentrate on explaining the technique. We are going to ignore some factors which must be taken into account in practical applications but which, in the present example, would only serve to clutter up the explanation. For example, the costs would be influenced by whether we use printed wiring, just how and what we are salvaging, whether special care should be taken with certain close tolerance components, the testing cost as a function of the limits, and so on.

With the above reservations in mind we make the following specifications for the example:

1. An L-C section will be the unit, the delay will be the response.

2. The raw cost of the unit will be the sum of the costs for the inductor and the capacitor plus a fixed cost K independent of the component tolerances. We will use K as a parameter.

3. The salvage value of a rejected unit will be one-half of the component costs.

4. All component distributions will be normal distributions about the nominal and will be in control.

We will consider two examples which differ from one another only in the tolerance on the over-all delay B, and for each example we will consider several different values for the fixed (i.e., independent of σ) cost K to be added to the component cost to get the total raw cost. We introduce these variations to give the reader a quantitative idea of the trends they induce. We use the values given in Table II.

We must first examine the distribution of the delay (response) of the individual L-C sections as a function of the component tolerance distributions. Normalizing the formula for the delay so that L is in μh, C is in $\mu\mu f$, and Δ, x are in mμsec,

$$\Delta = 100 + x = \sqrt{CL} \tag{27}$$

(where x is the deviation of the response from nominal). Linearizing (27) and using the ordinary linear propagation of error formula one finds that the variance of x is

$$\sigma^2 = \tfrac{1}{4}\,[\sigma_L^2 + \sigma_C^2], \tag{28}$$

where σ_L (in μh) is the standard deviation of L, σ_C(in $\mu\mu f$) is the standard deviation of C, and σ (in mμsec) is the standard deviation of x. One should satisfy himself that x is normally, or approximately normally, distributed with mean zero, or approximately zero, and variance as in (28), or approximately as in (28). We can do so by using the nonlinear propagation of error approximate formulas[6] for the range of combinations of distributions for the components, Table II-8. It turns out that (see Table III)

(a) the variance as given by (28) is negligibly different from the true variance ($\leqq 1$ per cent, see column 5, Table III),

(b) the coefficient of skewness, $\beta_1 = \mu_3^2/\sigma^6$ is small ($\leqq 3$ per cent for all combinations, <1 per cent for most combinations, column 6, Table III) and

(c) the coefficient of excess $\beta_2 = \mu_4/\sigma^4 \approx 3$, the standard for normal

TABLE II — VALUES FOR EXAMPLE

1. Nominal inductance:	$L_0 = 100\ \mu h$
2. Nominal capacitance:	$C_0 = 100\ \mu\mu f$
3. Nominal delay per unit:	$\Delta_0 = \sqrt{LC} = 0.1\ \mu\text{sec}$
4. Number of units per delay line:	$n = 10$
5. Nominal delay of delay line:	$n\Delta_0 = 1\ \mu\text{sec}$
6. Tolerance on delay line,	
First example:	$B = 5$ mμsec (0.5 per cent)
Second example:	$B = 15$ mμsec (1.5 per cent)
7. Assumed risk of out of tolerance delay line:	$\epsilon = 0.01$ per cent ($r = 3.89$)
8. Cost versus tolerance of Components:	

Code†	Inductors		Capacitors	
	σ_L in μh	Cost	σ_C in $\mu\mu f$	Cost
1	0.577 (1%)*	\$10.00	0.577 (1%)*	\$1.00
2	1.155 (2%)	5.00	1.155 (2%)	0.50
3	2.887 (5%)	2.00	2.887 (5%)	0.30
4	5.774 (10%)	1.00	5.774 (10%)	0.20
5	11.55 (20%)	0.90	11.55 (20%)	0.15

9. Fixed cost parameter to be added to component costs to get total raw cost, both examples:	K = \$0.30 1.00 3.00 5.00 10.00 15.00
10. Salvage value:	$\alpha C = \frac{1}{2}$ component cost for all σ.

* The figures in parenthesis give the tolerances for uniform distributions which have the same standard deviations as the normal distributions.

† We will use the code number to refer to these distributions. The same code number is used for both inductors and capacitors. For a pair of components we will use the code pair (i, j) where i, the first entry, is the code for the inductor distribution, and j, the second entry, is the code for the capacitor. For example the code pair (2,1) means that $\sigma_L = 1.155$ at a cost of \$5.00 per inductor and $\sigma_C = 0.577$ at a cost of \$1.00 per capacitor.

distributions, (column 7, Table III) for the whole range of combinations of distributions. However, one does get into some small difficulty with the mean. The non-linear formula for the average, after dropping terms which turn out to be negligible in this case, is

$$\begin{aligned} \text{ave } x &= \tfrac{1}{2}\,\Delta_{CC}\sigma_C^2 + \tfrac{1}{2}\,\Delta_{LL}\sigma_L^2, \\ &= -\tfrac{1}{8}\,10^{-2}(\sigma_C^2 + \sigma_L^2), \end{aligned} \tag{29}$$

where the partial derivatives are evaluated at $L = 100$, $C = 100$. One finds that

$$\frac{\text{ave } x}{\sigma} = \frac{\bar{x}}{\sigma} = -\frac{1}{4}\cdot 10^{-2}\sqrt{\sigma_L^2 + \sigma_C^2} \tag{30}$$

TABLE III

Various Indices of the Distribution of x as a Function of the Component Distributions. (For a normal distribution with mean zero and variance $\sigma^2 = \frac{1}{4}(\sigma_L^2 + \sigma_C^2)$ these indices have the values shown in the first row.)

Code	$-\frac{\bar{x}}{\sigma}$¶	$-\frac{10\bar{x}}{5}$¶	$-\frac{10\bar{x}}{15}$¶	$\frac{\delta\sigma^2}{\sigma^2}$	β_1†	β_2	Minimums of C*‡ at (√) for B = 5	Minimums of C*‡ at (√) for B = 15
$N(0,\sigma)$	0	0	0	0	0	3		
1,1	0.002	0.002	0.001	0.000+	0.000+	3	√	
1,2	0.003	0.004	0.001	0.000+	0.000+	3	√	
1,3	0.007	0.022	0.007	0.001	0.001	3		
1,4	0.015	0.084	0.028	0.003	0.007	3		
1,5	0.028	0.326	0.108	0.012	0.030	3		
2,2	0.004	0.007	0.002	0.000+	0.000+	3		
2,3	0.008	0.024	0.008	0.001	0.000+	3	√	√
2,4	0.015	0.088	0.029	0.003	0.006	3		
2,5	0.029	0.339	0.113	0.012	0.028	3		
3,3	0.010	0.042	0.014	0.001	0.000+	3		
3,4	0.016	0.104	0.034	0.002	0.001	3		√
3,5	0.031	0.357	0.119	0.011	0.020	3		
4,4	0.020	0.167	0.056	0.003	0.000+	3		
4,5	0.032	0.419	0.140	0.009	0.005	3		
5,5	0.041	0.671	0.224	0.010	0.000+	3		

¶ $\bar{x}$ = ave x.

† $\sqrt{\beta_1}$ is negative, i.e., negative skewness.

‡ Checks (√) in the last two columns indicate combinations which are used in final solutions for C^*. The left column is for $B = 5$, the right for $B = 15$.

is not significant (≤ 4 per cent for all combinations of σ_L, σ_C, and <2 per cent for all combinations not involving any code 5 element). However, now compute the ratio

$$\frac{n \text{ ave } x}{B} = \frac{10 \text{ ave } x}{B} = \frac{10\bar{x}}{B}, \tag{31}$$

which is a measure of the shift in the average of the delay for the complete delay line compared to the tolerance on the delay line. One finds this ratio is appreciable (i.e., about 10 per cent or more) for some of the combinations of component tolerances; namely, for $B = 5$, all combinations involving code 5 elements and the one other combination (4,4); for $B = 15$ only the combination (5,5). Thus, in general, the assumptions on normality and mean zero are approximately fulfilled; however, one should view with suspicion any solution we may get which involves one of the above mentioned combinations.*

* Note that these same combinations include all the larger values for β_1.

TABLE IV

Calculation of $C^*(\sigma)$ for K = \$0.30, B = 5 mμsec

Code	σ	$c(\sigma)$	q	$H(q)$	α	$C^*(\sigma)$
1,1	0.408	\$11.00	1.004	1.001±	0.49	\$11.30
2,1	0.646	6.00	1.59	1.30	0.48	7.30
2,2	0.817	5.50	2.01	1.56	0.47	7.50
3,1	1.47	3.00	3.62	2.68	0.45	6.35
3,2	1.55	2.50	3.81	2.82	0.45	5.60
3,3	2.04	2.30	5.02	3.64	0.44	6.45
4,2	2.94	1.50	7.24	5.23	0.42	6.20
4,3	3.23	1.30	7.95	5.70	0.41	6.05
4,4	4.08	1.20	10.04	7.5±	0.40	7.35
5,4	6.46	1.10	15.88	—	0.39	—
5,5	8.17	1.05	20.09	—	0.39	—

The next item we want is the raw cost function, $C(\sigma)$. From the formula for σ, (28), and from the component cost versus tolerance functions, Table II-8, we can compute the cost and the variance for every combination of tolerance distributions for L and C. These points are shown in Fig. 1; the numbers near each point are the code pairs which indicate the particular combination of distributions used to calculate each point. The points which are connected together, call them $c(\sigma)$, give the raw cost of the components (only) since this is the lower envelope of the set of all points. To get the raw cost we must add the fixed cost parameter K, i.e.,

$$C(\sigma) = K + c(\sigma). \tag{32}$$

The points which make up $c(\sigma)$ are tabulated in Table IV in the first three columns; the first column shows the particular combination of component distributions, the second column, σ, and the third column, c.

All that remains is to perform the calculations set forth in the text. We will perform these calculations carefully for one set of data in order to show the method. We will take the case $B = 5$ and $K = 0.30$; therefore,

$$q = \frac{3.89\sqrt{10}}{5}\sigma = 2.46\sigma, \tag{33}$$

and

$$C(\sigma) = 0.30 + c(\sigma). \tag{34}$$

The value of q for each σ is shown in the fourth column, Table IV. For each q we find the corresponding $H(q)$ from either Table I or Fig. 2, whichever is more convenient; $H(q)$ is entered in the fifth column,

Table IV. Since the salvage value is one-half the component cost

$$\alpha = \frac{c(\sigma)/2}{C(\sigma)} = \frac{c(\sigma)/2}{0.30 + c(\sigma)}; \tag{35}$$

α is entered in the sixth column, Table IV. The real cost, C^*, is to be computed from this data. Modifying (21) to fit the form of this data, C^* is given by

$$C^*(\sigma) = [(1 - \alpha)H(q) + \alpha][30 + c(\sigma)]; \tag{36}$$

C^* is entered in the last column, Table IV, and is plotted in Fig. 3 as the curve marked $K = \$0.30$. From either the tabular form of C^* or the graphical form of C^* it is easy to see that the minimum real cost is $C^* = \$5.60$ per unit. Checking back through the calculations we see

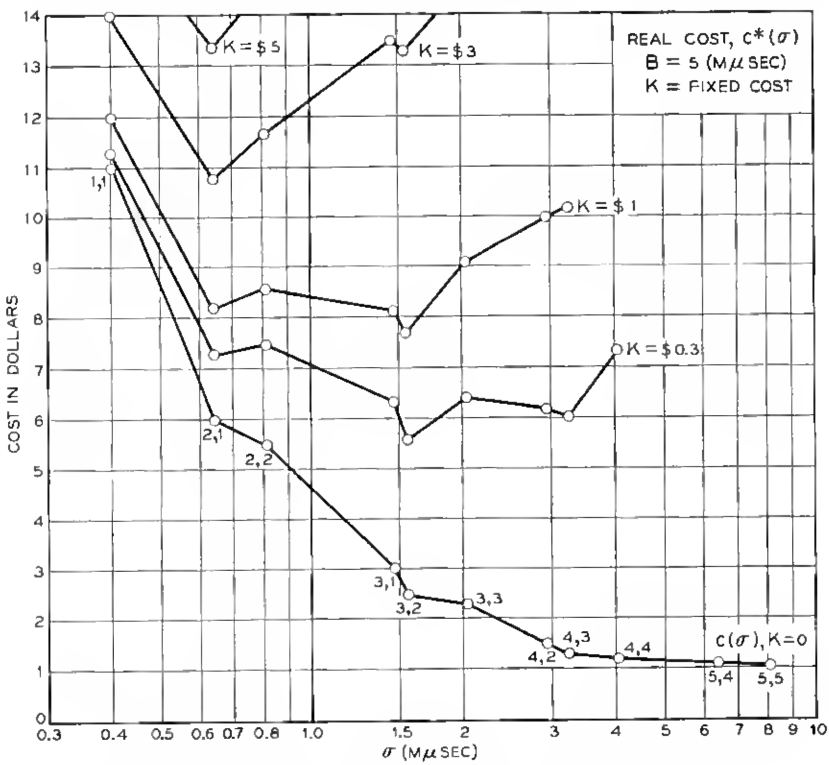

Fig. 3 — Real cost per section of a ten-section, one-microsecond, lumped-constant delay line as a function of the standard deviation of the delay per section and for various values of the fixed cost K. The delay tolerances are $\pm B = \pm 5$ mμsec.

that the minimum real cost is realized for the code pair (3,2), i.e., $\sigma_L = 2.89$ and $\sigma_C = 1.16$. The rejection limits, $\pm b$, are calculated from (22) and for this case

$$b = \sigma w(q) = 1.55\, w(q) = 0.72 \text{ m}\mu\text{sec} \tag{37}$$

where $w(q)$ is obtained from Table I or Fig. 2. Finally, the rejection rate, (23), is 65 per cent.

In addition to the case detailed above different values of the fixed cost were considered, as listed in Table II-9. The next variation was to change the over-all tolerance to $B = 15$ and consider the same range of values for K again. In all cases the salvage value was taken as one-half of the material cost, for simplicity.

Furthermore, all of the above cases were considered using the other

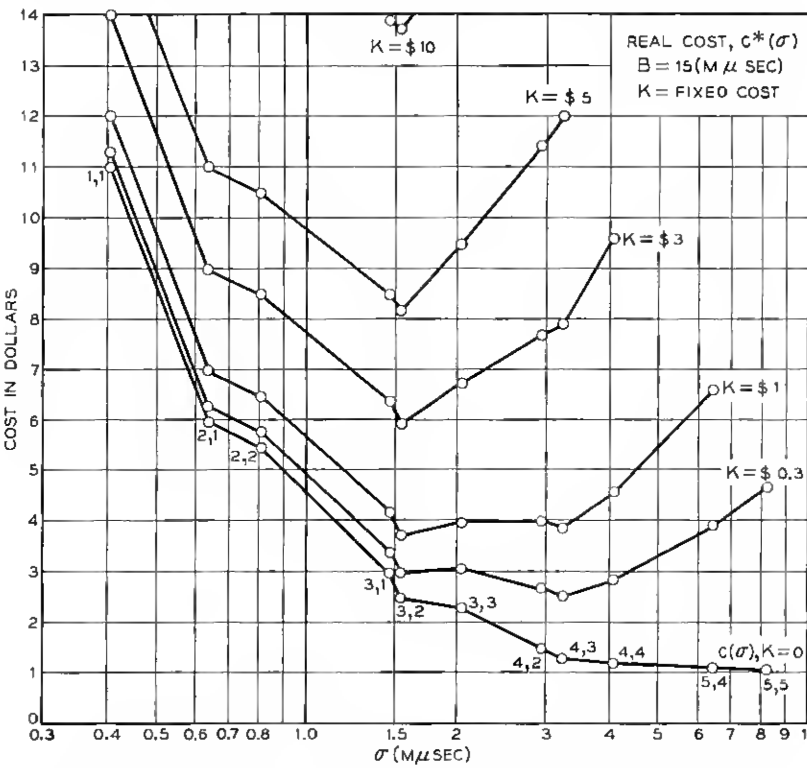

Fig. 4 — Real cost per section of a ten-section, one-microsecond, lumped-constant delay line as a function of the standard deviation of the delay per section and for various values of the fixed cost K. The delay tolerances are $\pm B = \pm 15$ mμsec.

TABLE V — RESULTS OF EXAMPLE

Line	Risk, ϵ, per cent	Fixed Cost, K	Min C^*	Code	Rejection Rate, per cent
		$B = 5$			
1	0.01	$0.30	$5.60	3,2	65
2	0.01	1.00	7.60	3,2	65
3	0.01	3.00	10.80	2,1	23
4	0.01	5.00	13.40	2,1	23
5	0.01	10.00	19.90	2,1	23
6	0.01	15.00	26.02	1,1	0.1
7	0.0	0.30	7.45	3,2	75
8	0.0	1.00	10.15	2,1	44
9	0.0	3.00	13.70	2,1	44
10	0.0	5.00	17.30	2,1	44
11	0.0	10.00	25.45	1,1	22
12	0.0	15.00	31.85	1,1	22
13	0.01+	min $C^* = 11.00 + K$		1,1	no-test
		$B = 15$			
14	0.01	$0.30	$2.55	4,3	50
15	0.01	1.00	3.75	3,2	10
16	0.01	3.00	5.95	3,2	10
17	0.01	5.00	8.20	3,2	10
18	0.01	10.00	13.75	3,2	10
19	0.01	15.00	19.30	3,2	10
20	0.0	0.30	3.30	4,3	64
21	0.0	1.00	4.65	3,2	34
22	0.0	3.00	7.65	3,2	34
23	0.0	5.00	10.05	2,2	2
24	0.0	10.00	16.25	2,1	2
25	0.0	15.00	21.35	2,1	2
26	$<10^{-4}$	min $C^* = 5.50 + K$		2,2	no-test
27	0.1	min $C^* = 3.00 + K$		3,1	no-test

criteria discussed previously. That is, the no-test method, (25), for $\epsilon = 0.01$ per cent, and the zero risk method, (26).

The results of the above calculations are shown in Figs. 3 and 4 and in Table V. The results of the zero-risk method were not plotted; the curves are similar to the ones shown but are shifted in a manner indicated by the shift of the minimums as recorded in Table V. Obviously, no curves of this type can be plotted for the no-test method.

VIII. DISCUSSION OF EXAMPLE

The author must admit that this is not the best of all possible examples. Better examples of units would be the amplifiers in a long transmission line, or individual logic packages in a logic network, or gas tube crosspoints in a switching network, and so forth. However, there is a very real difficulty involved in constructing such examples; namely, it is extremely difficult, but possible, to obtain the raw cost as a function

of the variance for any but the simplest kind of unit. Therefore, in order not to get side-tracked the author has chosen a simple unit, a delay network, and asks the reader to use his imagination. In addition to asking the reader to accept some of the simplifications and their likes previously noted, and to accept the possibility of making some difficult tests (e.g., rejecting on ± 0.7 mμsec, which, however, is certainly no more difficult than rejecting on ± 0.5 mμsec as would be necessary in the zero-risk case), he also asks the reader to imagine some good economic reason why the completed delay lines cannot be tested to determine whether they are within the tolerance limits. For, if the completed delay lines could be tested we would have to consider another possible production process.

It is to be emphasized, however, that all the shortcomings of this example can be overcome because most of the extensions we require are not out of line with usual practices, cf., parenthetical statement in the preceding paragraph, for example. If it would make the reader any happier he can relabel the scales in Fig. 1, the raw cost curve, and pretend that he has an amplifier, for instance. The calculations will be the same from there on except for any adjustments in the salvage values that the reader cares to make.

Probably the most striking result is the size of the rejection rate. For instance, in Table V, lines 1,2 the rejection rate is 65 per cent which is much larger than the rejection rates for production processes which are usually considered as satisfactory. The important point to realize, however, is that *under the assumptions* considered and in order to produce delay lines at the *minimum cost per usable delay line* this is the rejection rate. All rejection rates are not this high. For, as the fixed cost K is increased (lines 3, 4, 5) the rejection rate decreases to 23 per cent. Of course, a rejection rate of 23 per cent is also rather high compared to the usual. It is not until the fixed cost is increased to \$15 (line 6) that the rejection rate is of a usual size, about 0.1 per cent.

For the set of entries in Table III which correspond to $B = 15$ the rejection rates are usual (10 per cent for lines 2, 3, 4, 5, 6) except for line 1 (50 per cent).

For comparison we have also tabulated the minimum real costs for the zero-risk case and for the no-test case in Table V. These values were obtained under the same assumptions as were used above. But, in order to compare the real costs on a fair basis one must attach a cost figure to the risk of having one out of ten thousand delay lines out of tolerance, or to the cost of 100 per cent testing of the individual units, respectively. Note that for $B = 15$, the no-test case (lines 26, 27), we run into trouble because of component availability. The only compo-

nents which are available either make the risk much lower than desired — at a high cost — or they make the risk too high.

It is worthwhile making a point here. We assumed that we knew the raw cost at one stage in the procedure. On that assumption we obtained a minimum, or a set of minimums and near minimums, for the real cost. Also, we could obtain all the associated rejection limits. We now have some new knowledge which is important if the cost of testing is not insignificant compared to the raw cost. For, notice, it is certainly more expensive to accept or reject on, say, 0.72σ than on 2.0σ. And this is a type of information which we could not have used intelligently initially but which we can use now. Hence, we can now readjust the raw cost curve and reperform the calculations.

The above idea illustrates a general principle. One need not think of this procedure, as a whole, as necessarily leading to the answer in one stroke. Rather, one should think of it as a procedure which can be applied again and again in order to converge on the answer. For, after any one application the number of combinations of component tolerances which are candidates for the production model has been reduced, and one can then afford to get more precise information on fewer possibilities in order to reapply the method.

IX. CONCLUSIONS

We have given a method for finding the optimum tolerance assignment from the viewpoint of giving the lowest cost per acceptable unit. We have compared it with two other criteria in an idealized example and have shown that the method described is usually the best – for this example.

As was noted in the discussion of the example, the most startling result has been the generally large sizes of the rejection rates. In order to investigate this, one must remember that the rejection rate has been thrown in as another variable instead of allowing the rejection rate to be a measure of the optimization of the production method. Now a low rejection rate is not a bad measure for many production processes; the thing which these processes have in common is that the ratio of the material cost to the labor cost and other fixed costs is very small, assuming that these costs are unsalvageable.* This was the condition for some of the cases considered. And, in general, if this be the case the method we have given will also predict a low rejection rate since the raw cost will be almost a constant. However, the advantage our method has is

* These costs are not necessarily unsalvageable. One might often be able to salvage a good share of the labor cost by selectively assembling the rejected units.

that it will handle the other cases, too. These cases are not unusual today and with increasing automation they will become more and more the order of the day.

A word of caution is important. We have given a method of optimizing tolerances to reduce manufacturing costs under a special set of assumptions. *Our analysis and conclusions are valid only for manufacturing processes in which these assumptions prevail under actual manufacturing conditions.*

In obtaining our formulas we have considered only the special case of symmetric tolerance limits. It is obvious that the method can easily be extended to cover the case of one-sided tolerance limits and the case of unsymmetric tolerance limits. Further, we have only considered the case in which all the distributions are normal; it is not so obvious how the method can practicably be extended to the non-normal case. However, for distributions which are not violently non-normal it is not clear that what we have done is not of sufficient accuracy. Indeed, in practice, both the raw cost and salvage rate as functions of the variance of the response of the unit would be given only approximately. Further, it appears from the example that the minimum is not critical. Thus, the principal benefit of the method would be to get an approximation to the optimum. Then, after finding the correct neighborhood one could make an exhaustive cost and statistical study to determine the optimum production process.

Also, we have dealt only with the case where there is only one response per unit to be considered. The practicable extension of this work to multiple responses is not obvious. However, if one can single out one response which is more critical than the others remarks similar to the ones in the preceding paragraph about using this method to get in the correct neighborhood are in order.

X. ACKNOWLEDGMENT

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